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Some new results on the triple-humped soliton equation: Bäcklund transformation, superposition principle and new explicit solutions

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Abstract. A triple-humped soliton equation is considered. A corresponding Bäcklund transformation for it is obtained and a nonlinear superposition formula is presented. As an application of the results obtained, soliton solutions first found by Narita to the system are rederived. A sequence of rational solutions are found and other types of solutions which are a mix of exponentials and rational expressions are deduced.

1. Introduction

The so-called triple-humped soliton equation under consideration is [1, 2]

$$u_t(n) = 12\frac{(1-u(n+2)-u(n+1))^{1/3}(1-u(n)-u(n-1))^{1/3}}{(1-u(n+2)-u(n+1))^{1/3}+(1-u(n)-u(n-1))^{1/3}} -12\frac{(1-u(n+1)-u(n))^{1/3}(1-u(n-1)-u(n-2))^{1/3}}{(1-u(n+1)-u(n))^{1/3}+(1-u(n-1)-u(n-2))^{1/3}}. \qquad (1)$$

Some research on this equation has been conducted. For example in [1], the triple-humped one-soliton solution for (1) was found. Further in [2], by using the dependent variable transformation

$$u(n) = v(n+\tfrac{1}{2}) - v(n-\tfrac{1}{2})$$

$$v(n) = C - 6\frac{f_z(n)}{f(n)} - 8\left[\frac{f_z(n)}{f(n)}\right]^3 + 12\left[\frac{f_z(n)}{f(n)}\right]\left[\frac{f_{zz}(n)}{f(n)}\right] - 4\left[\frac{f_{zzz}(n)}{f(n)}\right] - 4\frac{g(n)}{f(n)}$$

with C a constant and z an auxiliary variable, Narita transformed (1) into the following bilinear equations

$$D_z(D_t + 2\sinh(2D_n))f(n)\cdot f(n) = 0 \qquad (2)$$

$$\sinh(D_n)[D_z\cosh(D_n) + \sinh(D_n)]f(n)\cdot f(n) = 0 \qquad (3)$$

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$$D_z[D_z\cosh(D_n)+\sinh(D_n)]f(n)\cdot f(n)=0 \tag{4}$$

$$D_z^3\sinh(D_n)f(n)\cdot f(n)-2\sinh(D_n)f(n)\cdot g(n)=0 \tag{5}$$

where the bilinear operators are defined as follows [3–5]

$$D_z^m D_t^k\ a\cdot b\equiv\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^k a(z,t)b(z',t')|_{z'=z,t'=t}$$

$$\exp(\delta D_n)a(n)\cdot b(n)\equiv\exp\left[\delta\left(\frac{\partial}{\partial n}-\frac{\partial}{\partial n'}\right)\right]a(n)b(n')|_{n'=n}=a(n+\delta)b(n-\delta).$$

N-soliton solutions for (2)–(5) and thus (1) were obtained.

The purpose of this paper is to present a Bäcklund transformation (BT) and nonlinear superposition formulae for (2)–(5). As an application of the results obtained, multisoliton solutions first obtained by Narita can be rederived. A sequence of rational solutions are obtained and other types of solutions which are a mix of exponentials and rational expressions are deduced.

This paper is organized as follows. Section 2 is devoted to deriving a BT for equations (2)–(5). In section 3, a nonlinear superposition formula is rigorously proved. Another nonlinear superposition formula is obtained in section 4. As a result, a sequence of rational solutions are generated. Another type of solutions are given in section 5. In section 6, a conclusion is given. Finally in the appendix, we list some bilinear operator identities which are used in this paper.

2. A Bäcklund transformation

In this section, we derive a BT for equations (2)–(5). The result obtained is the following.

Proposition 1. A Bäcklund transformation for (2)–(5) is

$$(D_z+\lambda^{-1}\mathrm{e}^{-2D_n}+\gamma)f(n)\cdot f'(n)=0 \tag{6}$$

$$\mathrm{e}^{-D_n}f(n)\cdot f'(n)=(-\lambda^{-1}\mathrm{e}^{-3D_n}+\mu\mathrm{e}^{D_n})f(n)\cdot f'(n) \tag{7}$$

$$(D_t\mathrm{e}^{-D_n}-\lambda\mathrm{e}^{D_n}+k\mathrm{e}^{-D_n})f(n)\cdot f'(n)=0 \tag{8}$$

$$(D_z+\lambda^{-1}D_z\mathrm{e}^{-2D_n}+\lambda^{-1}\gamma\mathrm{e}^{-2D_n}+\omega)f(n)\cdot f'(n)=0 \tag{9}$$

$$\begin{aligned}&[D_z^3+3\gamma D_z^2-3\lambda^{-1}D_z^2\mathrm{e}^{-2D_n}+(3\lambda^{-1}-6\lambda^{-1}\gamma)D_z\mathrm{e}^{-2D_n}+\gamma(3\lambda^{-1}-6\lambda^{-1}\gamma)\mathrm{e}^{-2D_n}+\nu]\\&\qquad\times f(n)\cdot f'(n)-f(n)g'(n)+f'(n)g(n)=0\end{aligned} \tag{10}$$

where $\lambda,\mu,\gamma,k,\omega$ and ν are arbitrary constants.

Proof. Let $(f(n),g(n))$ be a solution of equations (2)–(5). If we can show that equations (6)–(10) guarantee that the following four relations,

$$P_1\equiv D_z(D_t+2\sinh(2D_n))f'(n)\cdot f'(n)=0$$

$$P_2\equiv\sinh(D_n)[D_z\cosh(D_n)+\sinh(D_n)]f'(n)\cdot f'(n)=0$$

$$P_3\equiv D_z[D_z\cosh(D_n)+\sinh(D_n)]f'(n)\cdot f'(n)=0$$

$$P_4\equiv D_z^3\sinh(D_n)f'(n)\cdot f'(n)-2\sinh(D_n)f'(n)\cdot g'(n)=0$$

hold, then equations (6)–(10) form a BT, and $(f'(n),g'(n))$ given by (6)–(10) is a new solution of (2)–(5).

In fact, similar to the proof in [6–9], we know that $P_i = 0$ ($i = 1, 2, 3$) can be proved using equations (6)–(9). Thus, it suffices to show that $P_4 = 0$. For this, by making use of (A1)–(A6), (6)–(10), we have

$$
\begin{aligned}
-[\mathrm{e}^{D_n} f(n)\cdot f(n)]P_4 = {} & [D_z^3\sinh(D_n) f(n)\cdot f(n) - 2\sinh(D_n) f(n)\cdot g(n)] \\
& \times[\mathrm{e}^{D_n} f'(n)\cdot f'(n)] \\
& -[D_z^3\sinh(D_n) f'(n)\cdot f'(n) - 2\sinh(D_n) f'(n)\cdot g'(n)][\mathrm{e}^{D_n} f(n)\cdot f(n)] \\
& -3[D_z(D_z\cosh(D_n)+\sinh(D_n)) f(n)\cdot f(n)][D_z\mathrm{e}^{D_n} f'(n)\cdot f'(n)] \\
& +3[D_z(D_z\cosh(D_n)+\sinh(D_n)) f'(n)\cdot f'(n)][D_z\mathrm{e}^{D_n} f(n)\cdot f(n)] \\
= {} & 2\sinh(D_n)[(D_z^3 f(n)\cdot f'(n))\cdot f(n) f'(n) \\
& -3(D_z^2 f(n)\cdot f'(n))\cdot(D_z f(n)\cdot f'(n))] \\
& -2\sinh(D_n)(f(n)g'(n) - f'(n)g(n))\cdot f(n) f'(n) \\
= {} & 2\sinh(D_n)(D_z^3 f(n)\cdot f'(n))\cdot f(n) f'(n) \\
& -2\sinh(D_n)(f(n)g'(n) - f'(n)g(n))\cdot f(n) f'(n) \\
& +6\sinh(D_n)(D_z^2 f(n)\cdot f'(n))\cdot[(\lambda^{-1}\mathrm{e}^{-2D_n}+\gamma) f(n)\cdot f'(n)] \\
= {} & 2\sinh(D_n)[(D_z^3+3\gamma D_z^2) f(n)\cdot f'(n) + f'(n)g(n) - f(n)g'(n)]\cdot f(n) f'(n) \\
& +6\lambda^{-1}\{\sinh(D_n) f(n) f'(n)\cdot(D_z^2\mathrm{e}^{-2D_n} f(n)\cdot f'(n)) \\
& -D_z\cosh(D_n)[(D_z\mathrm{e}^{-2D_n} f(n)\cdot f'(n))\cdot f(n) f'(n) \\
& +(\mathrm{e}^{-2D_n} f(n)\cdot f'(n))\cdot(D_z f(n)\cdot f'(n))]\} \\
= {} & 2\sinh(D_n)[(D_z^3+3\gamma D_z^2-3\lambda^{-1}D_z^2\mathrm{e}^{-2D_n}) f(n)\cdot f'(n) \\
& + f'(n)g(n) - f(n)g'(n)]\cdot f(n) f'(n) \\
& +6D_z\cosh(D_n)[(\lambda^{-1}\gamma-\lambda^{-1})\mathrm{e}^{-2D_n} f(n)\cdot f'(n)]\cdot f(n) f'(n) \\
& +6\lambda^{-1}\gamma D_z\cosh(D_n)[\mathrm{e}^{-2D_n} f(n)\cdot f'(n)]\cdot f(n) f'(n) \\
= {} & 2\sinh(D_n)[(D_z^3+3\gamma D_z^2-3\lambda^{-1}D_z^2\mathrm{e}^{-2D_n}) f(n)\cdot f'(n) \\
& + f'(n)g(n) - f(n)g'(n)]\cdot f(n) f'(n) \\
& +6D_z\cosh(D_n)[(2\lambda^{-1}\gamma-\lambda^{-1})\mathrm{e}^{-2D_n} f(n)\cdot f'(n)]\cdot f(n) f'(n) \\
= {} & 2\sinh(D_n)[(D_z^3+3\gamma D_z^2-3\lambda^{-1}D_z^2\mathrm{e}^{-2D_n}) f(n)\cdot f'(n) \\
& + f'(n)g(n) - f(n)g'(n)]\cdot f(n) f'(n) \\
& +6(2\lambda^{-1}\gamma-\lambda^{-1})\sinh(D_n)[f(n) f'(n)\cdot(D_z\mathrm{e}^{-2D_n} f(n)\cdot f'(n)) \\
& -(D_z f(n)\cdot f'(n))\cdot(\mathrm{e}^{-2D_n} f(n)\cdot f'(n))] \\
= {} & 2\sinh(D_n)[(D_z^3+3\gamma D_z^2-3\lambda^{-1}D_z^2\mathrm{e}^{-2D_n} \\
& +3(\lambda^{-1}-2\lambda^{-1}\gamma)D_z\mathrm{e}^{-2D_n}) f(n)\cdot f'(n) \\
& + f'(n)g(n) - f(n)g'(n)]\cdot f(n) f'(n) \\
& -6(2\lambda^{-1}\gamma-\lambda^{-1})\gamma\sinh(D_n)(\mathrm{e}^{-2D_n} f(n)\cdot f'(n))\cdot f(n) f'(n) \\
= {} & 0.
\end{aligned}
$$

□

3. A nonlinear superposition formula

We shall simply denote, without confusion, $f(n,t) = f(n)$ or f. The result reached is the following.

Proposition 2. Let $(f_0(n), g_0(n))$ be a solution of equations (2)–(5) and suppose that $(f_i(n), g_i(n))$ $(i = 1, 2)$ are solutions of (2)–(5) which are related to $(f_0(n), g_0(n))$ under the BT equations (6)–(10) with parameters $(\lambda_i, \mu_i, \gamma_i, k_i, \omega_i, \nu_i)$, i.e. $(f_0, g_0) \stackrel{(\lambda_i,\mu_i,\gamma_i,k_i,\omega_i,\nu_i)}{\longrightarrow} (f_i, g_i)$ $(i = 1, 2)$, $\lambda_1\lambda_2 \neq 0$, $f_j \neq 0$ $(j = 0, 1, 2)$. Then (f_{12}, g_{12}) defined by

$$\exp(-D_n) f_0(n) \cdot f_{12}(n) = c[\lambda_1 \exp(-D_n) - \lambda_2 \exp(D_n)] f_1(n) \cdot f_2(n) \tag{11}$$

$$\begin{aligned}&\exp(-D_n)(f_0(n) \cdot g_{12}(n) + g_0(n) \cdot f_{12}(n)) \\ &\qquad = c[\lambda_1 \exp(-D_n) - \lambda_2 \exp(D_n)](f_1(n) \cdot g_2(n) + g_1(n) \cdot f_2(n))\end{aligned} \tag{12}$$

where c is an arbitrary nonzero constant, is a new solution which is related to (f_1, g_1) and (f_2, g_2) under the BT (6)–(10) with parameters $(\lambda_2, \mu_2, \gamma_2, k_2, \omega_2, \nu_2)$, $(\lambda_1, \mu_1, \gamma_1, k_1, \omega_1, \nu_1)$ respectively.

Proof. Analogous to the deduction in [6–9], we can show that

$$(D_z + \lambda_2^{-1}\mathrm{e}^{-2D_n} + \gamma_2) f_1 \cdot f_{12} = 0 \tag{13}$$

$$(D_z + \lambda_1^{-1}\mathrm{e}^{-2D_n} + \gamma_1) f_2 \cdot f_{12} = 0 \tag{14}$$

$$(D_t \mathrm{e}^{-D_n} - \lambda_2 \mathrm{e}^{D_n} + k_2 \mathrm{e}^{-D_n}) f_1 \cdot f_{12} = 0 \tag{15}$$

$$(D_t \mathrm{e}^{-D_n} - \lambda_1 \mathrm{e}^{D_n} + k_1 \mathrm{e}^{-D_n}) f_2 \cdot f_{12} = 0 \tag{16}$$

$$\mathrm{e}^{-D_n} f_1 \cdot f_{12} = (-\lambda_2^{-1}\mathrm{e}^{-3D_n} + \mu_2 \mathrm{e}^{D_n}) f_1 \cdot f_{12} \tag{17}$$

$$\mathrm{e}^{-D_n} f_2 \cdot f_{12} = (-\lambda_1^{-1}\mathrm{e}^{-3D_n} + \mu_1 \mathrm{e}^{D_n}) f_2 \cdot f_{12} \tag{18}$$

$$(D_z + \lambda_2^{-1} D_z \mathrm{e}^{-2D_n} + \lambda_2^{-1}\gamma_2 \mathrm{e}^{-2D_n} + \omega_2) f_1 \cdot f_{12} = 0 \tag{19}$$

$$(D_z + \lambda_1^{-1} D_z \mathrm{e}^{-2D_n} + \lambda_1^{-1}\gamma_1 \mathrm{e}^{-2D_n} + \omega_1) f_2 \cdot f_{12} = 0 \tag{20}$$

$$-D_z f_1 \cdot f_2 + (\gamma_1 - \gamma_2) f_1 f_2 - \frac{1}{c\lambda_1\lambda_2} \mathrm{e}^{-2D_n} f_0 \cdot f_{12} = 0 \tag{21}$$

$$-D_z f_1 \cdot f_2 + (\omega_1 - \omega_2) f_1 f_2 - \frac{1}{c\lambda_1\lambda_2} D_z \mathrm{e}^{-2D_n} f_0 \cdot f_{12} - \frac{\gamma_1 + \gamma_2}{c\lambda_1\lambda_2} \mathrm{e}^{-2D_n} f_0 \cdot f_{12} = 0. \tag{22}$$

Therefore, in order to prove proposition 2, it suffices to show that

$$\begin{aligned}&[D_z^3 + 3\gamma_2 D_z^2 - 3\lambda_2^{-1} D_z^2 \mathrm{e}^{-2D_n} + (3\lambda_2^{-1} - 6\lambda_2^{-1}\gamma_2) D_z \mathrm{e}^{-2D_n} \\ &\qquad + \gamma_2(3\lambda_2^{-1} - 6\lambda_2^{-1}\gamma_2)\mathrm{e}^{-2D_n} + \nu_2] f_1 \cdot f_{12} - f_1 g_{12} + f_{12} g_1 = 0\end{aligned} \tag{23}$$

$$\begin{aligned}&[D_z^3 + 3\gamma_1 D_z^2 - 3\lambda_1^{-1} D_z^2 \mathrm{e}^{-2D_n} + (3\lambda_1^{-1} - 6\lambda_1^{-1}\gamma_1) D_z \mathrm{e}^{-2D_n} \\ &\qquad + \gamma_1(3\lambda_1^{-1} - 6\lambda_1^{-1}\gamma_1)\mathrm{e}^{-2D_n} + \nu_1] f_2 \cdot f_{12} - f_2 g_{12} + f_{12} g_2 = 0.\end{aligned} \tag{24}$$

Since (f_1, g_1) and (f_2, g_2) are two solutions of equations (2)–(5), we have, by use of (A1)–(A9), (11)–(13), (21), (22) and $(f_0, g_0) \longrightarrow^{(\lambda_2,\mu_2,\gamma_2,k_2,\omega_2,\nu_2)} (f_2, g_2)$, that

$$\begin{aligned}0 = &[D_z^3 \sinh(D_n) f_1 \cdot f_1 - 2\sinh(D_n) f_1 \cdot g_1][\mathrm{e}^{D_n} f_2 \cdot f_2] \\ &-[D_z^3 \sinh(D_n) f_2 \cdot f_2 - 2\sinh(D_n) f_2 \cdot g_2][\mathrm{e}^{D_n} f_1 \cdot f_1] \\ &-3[D_z(D_z \cosh(D_n) + \sinh(D_n)) f_1 \cdot f_1][D_z \mathrm{e}^{D_n} f_2 \cdot f_2] \\ &+3[D_z(D_z \cosh(D_n) + \sinh(D_n)) f_2 \cdot f_2][D_z \mathrm{e}^{D_n} f_1 \cdot f_1]\end{aligned}$$

$$
\begin{aligned}
&= D_z^3(\mathrm{e}^{D_n} f_1 \cdot f_2)\cdot(\mathrm{e}^{-D_n} f_1 \cdot f_2) - 2\sinh(D_n)(f_1 g_2 - f_2 g_1)\cdot f_1 f_2\\
&= -\frac{1}{c\lambda_1} D_z^3(\mathrm{e}^{-D_n} f_0 \cdot f_{12})\cdot(\mathrm{e}^{D_n} f_1 \cdot f_2) - (\mathrm{e}^{D_n} f_1 \cdot f_2)(\mathrm{e}^{D_n} g_2 \cdot f_1)\\
&\quad +(\mathrm{e}^{D_n} f_2 \cdot f_1)(\mathrm{e}^{D_n} g_1 \cdot f_2) + (\mathrm{e}^{-D_n} f_1 \cdot f_2)(\mathrm{e}^{-D_n} g_2 \cdot f_1)\\
&\quad -(\mathrm{e}^{-D_n} f_2 \cdot f_1)(\mathrm{e}^{-D_n} g_1 \cdot f_2)\\
&= -\frac{1}{c\lambda_1}\mathrm{e}^{-D_n}[(D_z^3 f_0 \cdot f_2)\cdot f_1 f_{12} - f_0 f_2 \cdot (D_z^3 f_1 \cdot f_{12})\\
&\quad -3(D_z^2 f_0 \cdot f_2)\cdot(D_z f_1 \cdot f_{12}) + 3(D_z f_0 \cdot f_2)\cdot(D_z^2 f_1 \cdot f_{12})]\\
&\quad -(\mathrm{e}^{D_n} f_1 \cdot f_2)(\mathrm{e}^{D_n} g_2 \cdot f_1) + (\mathrm{e}^{D_n} f_2 \cdot f_1)(\mathrm{e}^{D_n} g_1 \cdot f_2)\\
&\quad +(\mathrm{e}^{-D_n} f_1 \cdot f_2)(\mathrm{e}^{-D_n} g_2 \cdot f_1) - (\mathrm{e}^{-D_n} f_2 \cdot f_1)(\mathrm{e}^{-D_n} g_1 \cdot f_2)\\
&= -\frac{1}{c\lambda_1}\mathrm{e}^{-D_n}[(D_z^3 f_0 \cdot f_2)\cdot f_1 f_{12} - f_0 f_2 \cdot (D_z^3 f_1 \cdot f_{12})]\\
&\quad -\frac{3}{c\lambda_1}\mathrm{e}^{-D_n}[(D_z^2 f_0 \cdot f_2)\cdot((\lambda_2^{-1}\mathrm{e}^{-2D_n} + \gamma_2) f_1 \cdot f_{12})\\
&\quad -((\lambda_2^{-1}\mathrm{e}^{-2D_n} + \gamma_2) f_0 \cdot f_2)\cdot(D_z^2 f_1 \cdot f_{12})]\\
&\quad +\frac{1}{\lambda_1}(\mathrm{e}^{-D_n} g_2 \cdot f_1)[(\lambda_1 \exp(-D_n) - \lambda_2 \exp(D_n)) f_1 \cdot f_2]\\
&\quad -\frac{1}{\lambda_1}(\mathrm{e}^{D_n} f_1 \cdot f_2)[(\lambda_1 \exp(-D_n) - \lambda_2 \exp(D_n))(f_1 \cdot g_2 + g_1 \cdot f_2)]\\
&\quad +\frac{1}{\lambda_1}(\mathrm{e}^{-D_n} f_2 \cdot g_1)[(\lambda_1 \exp(-D_n) - \lambda_2 \exp(D_n)) f_1 \cdot f_2]\\
&= -\frac{1}{c\lambda_1}\mathrm{e}^{-D_n}\{[(D_z^3 + 3\gamma_2 D_z^2) f_0 \cdot f_2]\cdot f_1 f_{12} - f_0 f_2 \cdot [(D_z^3 + 3\gamma_2 D_z^2) f_1 \cdot f_{12}]\}\\
&\quad -\frac{3}{c\lambda_1\lambda_2}\{-\mathrm{e}^{-D_n}[(D_z^2 \mathrm{e}^{-2D_n} f_0 \cdot f_2)\cdot f_1 f_{12} - f_0 f_2 \cdot (D_z^2 \mathrm{e}^{-2D_n} f_1 \cdot f_{12})]\\
&\quad +2D_z \cosh(D_n)[(D_z \mathrm{e}^{-2D_n} f_0 \cdot f_{12})\cdot f_1 f_2 + (\mathrm{e}^{-2D_n} f_0 \cdot f_{12})\cdot(D_z f_1 \cdot f_2)]\}\\
&\quad +\frac{1}{c\lambda_1}[\mathrm{e}^{-D_n}(g_2 \cdot f_1 + f_2 \cdot g_1)][\mathrm{e}^{-D_n} f_0 \cdot f_{12}]\\
&\quad -\frac{1}{c\lambda_1}(\mathrm{e}^{D_n} f_1 \cdot f_2)[\mathrm{e}^{-D_n}(f_0 \cdot g_{12} + g_0 \cdot f_{12})]\\
&= -\frac{1}{c\lambda_1}\mathrm{e}^{-D_n}\{[(D_z^3 + 3\gamma_2 D_z^2 - 3\lambda_2^{-1} D_z^2 \mathrm{e}^{-2D_n}) f_0 \cdot f_2 - f_0 g_2 + f_2 g_0]\cdot f_1 f_{12}\\
&\quad - f_0 f_2 \cdot [(D_z^3 + 3\gamma_2 D_z^2 - 3\lambda_2^{-1} D_z^2 \mathrm{e}^{-2D_n}) f_1 \cdot f_{12} - f_1 g_{12} + f_{12} g_1]\}\\
&\quad +6\frac{\gamma_1 + \gamma_2 - 1}{c\lambda_1\lambda_2} D_z \cosh(D_n)(\mathrm{e}^{-2D_n} f_0 \cdot f_{12})\cdot f_1 f_2\\
&\quad -\frac{6}{c\lambda_1\lambda_2} D_z \cosh(D_n)(\mathrm{e}^{-2D_n} f_0 \cdot f_{12})\cdot(\gamma_1 - \gamma_2) f_1 f_2\\
&= -\frac{1}{c\lambda_1}\mathrm{e}^{-D_n}\{[(D_z^3 + 3\gamma_2 D_z^2 - 3\lambda_2^{-1} D_z^2 \mathrm{e}^{-2D_n}) f_0 \cdot f_2 - f_0 g_2 + f_2 g_0]\cdot f_1 f_{12}\\
&\quad - f_0 f_2 \cdot [(D_z^3 + 3\gamma_2 D_z^2 - 3\lambda_2^{-1} D_z^2 \mathrm{e}^{-2D_n}) f_1 \cdot f_{12} - f_1 g_{12} + f_{12} g_1]\}\\
&\quad +3\frac{2\gamma_2 - 1}{c\lambda_1\lambda_2}\mathrm{e}^{-D_n}[(D_z \mathrm{e}^{-2D_n} f_0 \cdot f_2)\cdot f_1 f_{12} + (D_z f_0 \cdot f_2)\cdot(\mathrm{e}^{-2D_n} f_1 \cdot f_{12})
\end{aligned}
$$

$$
\begin{aligned}
&-(e^{-2D_n} f_0 \cdot f_2) \cdot (D_z f_1 \cdot f_{12}) - f_0 f_2 \cdot (D_z e^{-2D_n} f_1 \cdot f_{12})] \\
=& -\frac{1}{c\lambda_1} e^{-D_n} \{[(D_z^3 + 3\gamma_2 D_z^2 - 3\lambda_2^{-1} D_z^2 e^{-2D_n} \\
&+3\lambda_2^{-1}(1-2\gamma_2) D_z e^{-2D_n}) f_0 \cdot f_2 - f_0 g_2 + f_2 g_0] \cdot f_1 f_{12} \\
&- f_0 f_2 \cdot [(D_z^3 + 3\gamma_2 D_z^2 - 3\lambda_2^{-1} D_z^2 e^{-2D_n} \\
&+3\lambda_2^{-1}(1-2\gamma_2) D_z e^{-2D_n}) f_1 \cdot f_{12} - f_1 g_{12} + f_{12} g_1]\} \\
&+3\frac{2\gamma_2 - 1}{c\lambda_1\lambda_2} e^{-D_n}[-\gamma_2 f_0 f_2 \cdot (e^{-2D_n} f_1 \cdot f_{12}) + (e^{-2D_n} f_0 \cdot f_2) \cdot \gamma_2 f_1 f_{12}] \\
=& \frac{1}{c\lambda_1} e^{-D_n} f_0 f_2 \cdot \{[D_z^3 + 3\gamma_2 D_z^2 - 3\lambda_2^{-1} D_z^2 e^{-2D_n} + 3\lambda_2^{-1}(1-2\gamma_2) D_z e^{-2D_n} \\
&+3\lambda_2^{-1}(1-2\gamma_2)\gamma_2 e^{-2D_n} + \nu_2] f_1 \cdot f_{12} - f_1 g_{12} + f_{12} g_1\} \\
=& 0
\end{aligned}
$$

which implies that (23) holds. Similarly we can prove (24) also holds. □

Using BT (6)–(10) and nonlinear superposition formula (11), (12), we can easily rederive N-soliton solutions of the equations (2)–(5). For example, by applying the BT (6)–(10) to the trivial solution $(f(n), g(n)) = (1, 1)$, we can easily obtain the one-soliton solution

$$(f'(n), g'(n)) = (1 + e^{\eta}, A e^{\eta})$$

where $\eta = 2(pn - \frac{1}{2}\tanh(2p)z - \sinh(4p)t + \eta^0)$, $A = \tanh^3(2p)$ and p, η^0 are constants, for the parameters

$$
\begin{aligned}
&\lambda = -(1 + e^{4p}) \qquad \mu = 1 + \lambda^{-1} \qquad \gamma = -\lambda^{-1} \\
&k = \lambda \qquad \omega = \lambda^{-2} \qquad \nu = 6\lambda^{-3} + 3\lambda^{-2} - 1.
\end{aligned}
$$

4. Nonlinear superposition formula and rational solutions

We now consider rational solutions of (1) or polynomial solutions of (2)–(5). In order to obtain polynomial solutions of (2)–(5), it is enough to consider special Bäcklund parameters of (6)–(10), i.e. $\lambda = -2$, $\mu = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $k = -2$, $\omega = \frac{1}{4}$, $\nu = 0$. In this case, (6)–(10) become

$$(D_z - \tfrac{1}{2}e^{-2D_n} + \tfrac{1}{2}) f(n) \cdot f'(n) = 0 \tag{25}$$

$$e^{-D_n} f(n) \cdot f'(n) = (\tfrac{1}{2}e^{-3D_n} + \tfrac{1}{2}e^{D_n}) f(n) \cdot f'(n) \tag{26}$$

$$(D_t e^{-D_n} + 2e^{D_n} - 2e^{-D_n}) f(n) \cdot f'(n) = 0 \tag{27}$$

$$(D_z - \tfrac{1}{2}D_z e^{-2D_n} - \tfrac{1}{4}e^{-2D_n} + \tfrac{1}{4}) f(n) \cdot f'(n) = 0 \tag{28}$$

$$(D_z^3 + \tfrac{3}{2}D_z^2 + \tfrac{3}{2}D_z^2 e^{-2D_n}) f(n) \cdot f'(n) - f(n)g'(n) + f'(n)g(n) = 0. \tag{29}$$

We shall represent the transformation (25)–(29) symbolically by $(f(n), g(n)) \longrightarrow (f'(n), g'(n))$. Now let $(f_0(n), g_0(n)), (f_1(n), g_1(n))$ and $(f_{12}(n), g_{12}(n))$ be three solutions of (2)–(5) and $(f_0(n), g_0(n)) \longrightarrow (f_1(n), g_1(n)) \longrightarrow (f_{12}(n), g_{12}(n))$, with $f_0(n), f_1(n), f_{12}(n) \neq 0$. Suppose that $f_2(n)$ is given by

$$\exp(-\tfrac{1}{2}D_n) f_0 \cdot f_{12} = c \sinh(\tfrac{1}{2}D_n) f_1 \cdot f_2 \qquad (c \text{ is a nonzero constant}). \tag{30}$$

From these assumptions and by use of (30) and Hirota bilinear operator identities (A10)–(A18), we can obtain

$$c D_z f_1(n) \cdot f_2(n) + e^{-2D_n} f_0(n) \cdot f_{12}(n) = c_1(t, z) f_1^2(n) \tag{31}$$

$$cD_t f_1(n)\cdot f_2(n) + 4f_0(n)f_{12}(n) = c_2(t,z)f_1^2(n) \tag{32}$$

$$\mathrm{e}^{-2D_n} f_0(n)\cdot f_{12}(n) - \frac{c}{2}\sinh(2D_n)f_1(n)\cdot f_2(n) = c_3(t,z)\mathrm{e}^{-2D_n} f_1(n)\cdot f_1(n) \tag{33}$$

$$D_z\mathrm{e}^{-2D_n} f_0(n)\cdot f_{12}(n) = c_4(t,z)f_1^2(n) \tag{34}$$

where $c_i(t,z)$ $(i = 1, 2, 3, 4)$ are suitable functions of t and z. Furthermore we assume that $f_2(n)$ determined by (30) is chosen such that $c_i(t,z) = 0$, $i = 1, 2, 3$ and $f_0(n)$, $f_{12}(n)$ are particularly chosen such that $c_4(t,z) = 0$. In this case, we have

$$cD_z f_1(n)\cdot f_2(n) + \mathrm{e}^{-2D_n} f_0(n)\cdot f_{12}(n) = 0 \tag{35}$$

$$cD_t f_1(n)\cdot f_2(n) + 4f_0(n)f_{12}(n) = 0 \tag{36}$$

$$\mathrm{e}^{-2D_n} f_0(n)\cdot f_{12}(n) - \frac{c}{2}\sinh(2D_n)f_1(n)\cdot f_2(n) = 0 \tag{37}$$

$$D_z\mathrm{e}^{-2D_n} f_0(n)\cdot f_{12}(n) = 0. \tag{38}$$

By using (30), (35)–(38), we can deduce that

$$(D_z - \tfrac{1}{2}\mathrm{e}^{-2D_n} + \tfrac{1}{2})f_0(n)\cdot f_2(n) = 0 \tag{39}$$

$$\mathrm{e}^{-D_n} f_0(n)\cdot f_2(n) = [\tfrac{1}{2}\mathrm{e}^{-3D_n} + \tfrac{1}{2}\mathrm{e}^{D_n}]f_0(n)\cdot f_2(n) \tag{40}$$

$$(D_t\mathrm{e}^{-D_n} + 2\mathrm{e}^{D_n} - 2\mathrm{e}^{-D_n})f_0(n)\cdot f_2(n) = 0 \tag{41}$$

$$(D_z - \tfrac{1}{2}D_z\mathrm{e}^{-2D_n} - \tfrac{1}{4}\mathrm{e}^{-2D_n} + \tfrac{1}{4})f_0(n)\cdot f_2(n) = 0. \tag{42}$$

Finally, we choose $g_2(n)$ as follows:

$$g_2(n) = \frac{(D_z^3 + \frac{3}{2}D_z^2 + \frac{3}{2}D_z^2\mathrm{e}^{-2D_n})f_0(n)\cdot f_2(n) + f_2(n)g_0(n)}{f_0(n)}. \tag{43}$$

Therefore, $(f_2(n), g_2(n))$ so obtained is a new solution of (2)–(5) and $(f_0(n), g_0(n)) \longrightarrow (f_2(n), g_2(n))$. Besides, we can show that $(f_2(n), g_2(n)) \longrightarrow (f_{12}(n), g_{12}(n))$.

To summarize, we can seek particular solutions of (2)–(5) via the following steps. First, choose a given solution $(f_1(n), g_1(n))$ of (2)–(5). Second, from the BT (25)–(29) we find $(f_0(n), g_0(n))$ and $(f_{12}(n), g_{12}(n))$ such that

$$(f_0(n), g_0(n)) \longrightarrow (f_1(n), g_1(n)) \longrightarrow (f_{12}(n), g_{12}(n))$$
$$D_z\mathrm{e}^{-2D_n} f_0(n)\cdot f_{12}(n) = 0$$

and, furthermore, obtain a particular solution $\tilde{f}_2(n)$ from (30). Then, a general solution of (30) is $f_2(n) = \tilde{f}_2(n) + J(t,z)f_1(n)$, where $J(t,z)$ is an arbitrary function of t, z. Finally, we substitute $f_2(n)$ into (31)–(33). If $J(t,z)$ can be determined such that $c_i(t,z) = 0$, $i = 1, 2, 3, 4$, the corresponding $(f_2(n), g_2(n))$ is a new solution of (2)–(5) where $g_2(n)$ is given by (43).

As an application of this result, we can obtain a sequence of polynomial solutions of (2)–(5). For example, if we choose $f_0(n) = g_0(n) = n - z - 4t + \alpha + 4$, $f_1(n) = g_1(n) = 1$ and $f_{12}(n) = g_{12}(n) = n - z - 4t + \alpha$ with α being a constant, then it is easily verified that $(n-z-4t+\alpha+4, n-z-4t+\alpha+4)$, $(n-z-4t+\alpha, n-z-4t+\alpha)$ and $(1, 1)$ are solutions of (2)–(5) and $(n-z-4t+\alpha+4, n-z-4t+\alpha+4) \longrightarrow (1, 1) \longrightarrow (n-z-4t+\alpha, n-z-4t+\alpha)$. Then we seek a solution in the form

$$f_2(n) = (n - z - 4t)^3 + A_1(t,z)(n - z - 4t)^2 + A_2(t,z)(n - z - 4t) + A_3(t,z)$$

such that (33)–(36) hold. A direct calculation shows that

$$c = -\tfrac{1}{3} \qquad A_1 = 3(\alpha + 2) \qquad A_2 = 3\alpha^2 + 12\alpha + 8 \qquad A_3 = 32t - 4z + c_0$$

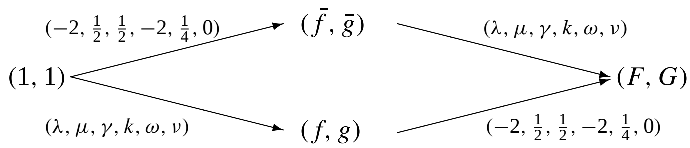
where c_0 is an arbitrary constant. The corresponding $g_2(n)$ is obtained from (41) as follows:

$$g_2(n) = f_2(n) - 12.$$

In this way, we may deduce a sequence of polynomial solutions of (2)–(5) and therefore the rational solutions of (1).

5. Other types of solutions

In this section, by combining the results in previous sections, we will deduce other types of solutions for (2)–(5) which are a mix of exponentials and rational expressions. As a simplest example, take a one-soliton solution and polynomial solution of (2)–(5), i.e. $(f, g) = (1 + e^{\eta}, Ae^{\eta})$, $(\bar{f}, \bar{g}) = (n - z - 4t + \alpha, n - z - 4t + \alpha)$. We have



where

$$F = -2(n - z - 4t + \alpha - 2)(1 + e^{\eta}) + (1 + e^{4p})(n - z - 4t + \alpha)(1 + e^{\eta - 4p})$$

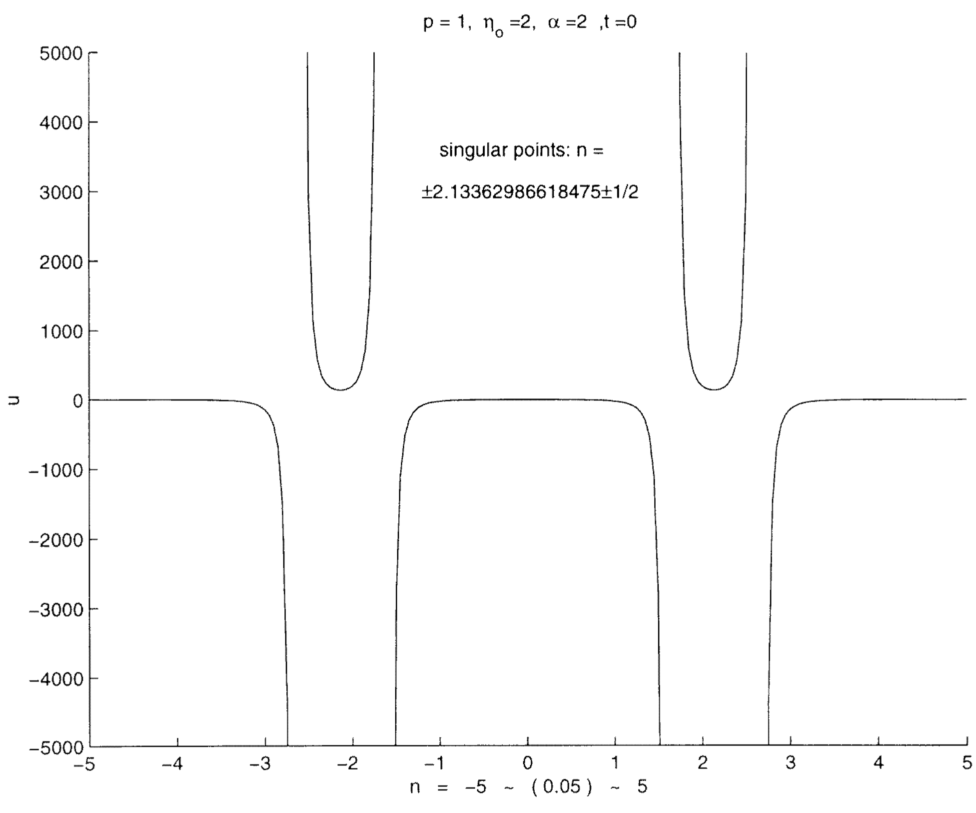


Figure 1. Profile of the solution (44) with (45).

$G = A\mathrm{e}^{\eta}[(\mathrm{e}^{-4p} - 1)(n - z - 4t + \alpha) + 4]$

with $\eta = 2(pn - \frac{1}{2}\tanh(2p)z - \sinh(4p)t + \eta^0)$, $A = \tanh^3(2p)$ and p, η^0 are constants, and

$$\lambda = -(1 + \mathrm{e}^{4p}) \qquad \mu = 1 + \lambda^{-1} \qquad \gamma = -\lambda^{-1}$$
$$k = \lambda \qquad \omega = \lambda^{-2} \qquad \nu = 6\lambda^{-3} + 3\lambda^{-2} - 1.$$

Thus, a particular solution of (1) is

$$u(n) = v(n + \tfrac{1}{2}) - v(n - \tfrac{1}{2}) \tag{44}$$

with

$$v(n) = \left\{ -6\frac{F_z(n)}{F(n)} - 8\left[\frac{F_z(n)}{F(n)}\right]^3 + 12\left[\frac{F_z(n)}{F(n)}\right]\left[\frac{F_{zz}(n)}{F(n)}\right] - 4\left[\frac{F_{zzz}(n)}{F(n)}\right] - 4\frac{G(n)}{F(n)} \right\}\Bigg|_{z=0}. \tag{45}$$

The plot of equation (44) with (45) is shown in figure 1.

Remark. The above superposition procedure could continue to generate more solutions of such type by purely algebraic means, although the corresponding calculations become more involved.

6. Summary

The so-called triple-humped soliton equation has been considered. Based on the bilinear forms of the equation by Narita, a corresponding BT for it is obtained. Furthermore, a nonlinear superposition formula is presented. As an application of the obtained results, soliton solutions first found by Narita to the system are rederived. A sequence of rational solutions are found and other types of solutions which are a mix of exponentials and rational expressions are also deduced.

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Appendix. Hirota bilinear operator identities

The following bilinear operator identities hold for arbitrary functions a–d.

$$\begin{aligned}(D_t^3\mathrm{e}^{D_n}a\cdot a)(\mathrm{e}^{D_n}b\cdot b) &- (\mathrm{e}^{D_n}a\cdot a)(D_t^3\mathrm{e}^{D_n}b\cdot b)\\ &-3(D_t^2\mathrm{e}^{D_n}a\cdot a)(D_t\mathrm{e}^{D_n}b\cdot b) + 3(D_t\mathrm{e}^{D_n}a\cdot a)(D_t^2\mathrm{e}^{D_n}b\cdot b)\\ &= 2\sinh(D_n)[(D_t^3a\cdot b)\cdot ab - 3(D_t^2a\cdot b)\cdot(D_ta\cdot b)]\\ &= D_t^3(\mathrm{e}^{D_n}a\cdot b)\cdot(\mathrm{e}^{-D_n}a\cdot b)\end{aligned} \tag{A1}$$

$$[\sinh(D_n)a\cdot b][\mathrm{e}^{D_n}c\cdot c] - [\sinh(D_n)c\cdot d][\mathrm{e}^{D_n}a\cdot a] = \sinh(D_n)(ad - cb)\cdot ac \tag{A2}$$

$$D_t \cosh(D_n)[(D_t e^{-2D_n} a \cdot b) \cdot ab + (e^{-2D_n} a \cdot b) \cdot (D_t a \cdot b)]$$
$$= \sinh(D_n)[ab \cdot (D_t^2 e^{-2D_n} a \cdot b) - (D_t^2 a \cdot b) \cdot (e^{-2D_n} a \cdot b)] \quad \text{(A3)}$$

$$D_t \cosh(D_n)(e^{-2D_n} a \cdot b) \cdot ab = \sinh(D_n)[ab \cdot (D_t e^{-2D_n} a \cdot b) - (D_t a \cdot b) \cdot (e^{-2D_n} a \cdot b)] \quad \text{(A4)}$$

$$D_t \cosh(D_n) a \cdot a = 0 \quad \text{(A5)}$$

$$\sinh(D_n) a \cdot a = 0 \quad \text{(A6)}$$

$$D_t^3 (e^{-D_n} a \cdot b) \cdot (e^{D_n} c \cdot d) = e^{-D_n}[(D_t^3 a \cdot d) \cdot cb - ad \cdot (D_t^3 c \cdot b)$$
$$-3(D_t^2 a \cdot d) \cdot (D_t c \cdot b) + 3(D_t a \cdot d) \cdot (D_t^2 c \cdot b)] \quad \text{(A7)}$$

$$2D_t \cosh(D_n)[(D_t e^{-2D_n} a \cdot b) \cdot cd + (e^{-2D_n} a \cdot b) \cdot (D_t c \cdot d)]$$
$$= e^{-D_n}[(D_t^2 e^{-2D_n} a \cdot d) \cdot cb] + (D_t^2 a \cdot d) \cdot (e^{-2D_n} c \cdot b)$$
$$-(e^{-2D_n} a \cdot d) \cdot (D_t^2 c \cdot b) - ad \cdot (D_t^2 e^{-2D_n} c \cdot b)] \quad \text{(A8)}$$

$$2D_t \cosh(D_n)(e^{-2D_n} a \cdot b) \cdot cd = e^{-D_n}[(D_t e^{-2D_n} a \cdot d) \cdot cb] + (D_t a \cdot d) \cdot (e^{-2D_n} c \cdot b)$$
$$-(e^{-2D_n} a \cdot d) \cdot (D_t c \cdot b) - ad \cdot (D_t e^{-2D_n} c \cdot b)] \quad \text{(A9)}$$

$$\sinh(D_n)(D_t a \cdot b) \cdot a^2 = D_t[\sinh(D_n) a \cdot b] \cdot [e^{D_n} a \cdot a] \quad \text{(A10)}$$

$$D_t[e^{-D_n} a \cdot b] \cdot [e^{D_n} c \cdot c] = e^{-D_n}[(D_t a \cdot c) \cdot cb - ac \cdot (D_t c \cdot b)]$$
$$= (D_t e^{-D_n} a \cdot c)(e^{-D_n} c \cdot b) - (e^{-D_n} a \cdot c)(D_t e^{-D_n} c \cdot b) \quad \text{(A11)}$$

$$2\sinh(D_n)[e^{-2D_n} a \cdot b] \cdot c^2 = e^{-D_n}[ac \cdot (e^{-2D_n} c \cdot b) - (e^{-2D_n} a \cdot c) \cdot cb] \quad \text{(A12)}$$

$$2\sinh(D_n) ab \cdot c^2 = (e^{D_n} a \cdot c)(e^{-D_n} c \cdot b) - (e^{-D_n} a \cdot c)(e^{D_n} c \cdot b) \quad \text{(A13)}$$

$$2\sinh(\delta_1 D_n)[e^{\delta_2 D_n} a \cdot b] \cdot [e^{\delta_2 D_n} c \cdot c] = [e^{(\delta_1+\delta_2) D_n} a \cdot c][e^{(\delta_2-\delta_1) D_n} c \cdot b]$$
$$-[e^{(\delta_2-\delta_1) D_n} a \cdot c][e^{(\delta_1+\delta_2) D_n} c \cdot b] \quad \text{(A14)}$$

$$2\sinh(D_n)[D_t e^{-2D_n} a \cdot b] \cdot c^2 = e^{-D_n}[(D_t a \cdot c) \cdot (e^{-2D_n} c \cdot b) + ac \cdot (D_t e^{-2D_n} c \cdot b)]$$
$$-e^{-D_n}[(D_t e^{-2D_n} a \cdot c) \cdot cb + (e^{-2D_n} a \cdot c) \cdot (D_t c \cdot b)] \quad \text{(A15)}$$

$$(D_t a \cdot b)c - (D_t a \cdot c)b = -a D_t b \cdot c \quad \text{(A16)}$$

$$[D_z e^{-D_n} a \cdot b][e^{-D_n} c \cdot d] - [e^{-D_n} a \cdot b][D_z e^{-D_n} c \cdot d] = D_z[e^{-D_n} a \cdot d] \cdot [e^{-D_n} c \cdot b] \quad \text{(A17)}$$

$$[e^{\delta D_n} a \cdot b][e^{-\delta D_n} c \cdot d] = e^{\delta D_n} ad \cdot cb. \quad \text{(A18)}$$

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